How can concepts be formalized?

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Slides in part based on

- Gerd Stumme
  - Course at Otto-von-Guericke Universität Magdeburg / Summer Term 2003
  - ECML PKDD Tutorial
- Rudolf Wille

Further Literature:
http://www.aifb.uni-karlsruhe.de/WBS/gst/FBA03/chapter1_2.pdf (Ganter / Stumme)
Overview

Today ‘s Agenda:

Categorization & Formal Concept Analysis

• Formal Context
• Formal Concepts
• Formal Concept Lattices
• FCA Implications
• Constructing Concept Lattices
Intension (Meaning)
• The specification of those qualities that a thing must have to be a member of the class

Extension (the objects in the class)
• Things that have those qualities
Categorization
[Mervis Rosch 1981]

Six salient problems:

• **Arbitrariness of categories.** Are there any a priori reasons for dividing objects into categories, or is this division initially arbitrary?

• **Equivalence of category members.** Are all category members equally representative of the category as has often been assumed?

• **Determinacy of category membership and representation.** Are categories specified by necessary and sufficient conditions for membership? Are boundaries of categories well defined?

• **The nature of abstraction.** How much abstraction is required— that is, do we need only memory for individual exemplars to account for categorization? Or, at the other extreme, are higher-order abstractions of general knowledge, beyond the individual categories, necessary?

• **Decomposability of categories into elements.** Does a reasonable explanation of objects consist in their decomposition into elementary qualities?

• **The nature of attributes.** What are the characteristics of these "attributes" into which categories are to be decomposed?
Formal Concept Analysis

Running Example:

Taste: Sweet/Sour, Shape: Round/Long/, Color: Red/Yellow/.., Texture: Smooth/Bumpy,

Ich möchte nur darauf hinweisen, daß es eine Zeit gab, in der man die Ähnlichkeit der
Empfindungen zur Basis der Kategorisierung von Pflanze und Tier gemacht hat. Man denke
[...] an die frühen Taxonomien des Ulisse Aldrovandi aus dem 16. Jahrhundert, der die
scheußlichen Tiere (die Spinnen, Molche und Schlagen) und die Schönheiten (die
Leoparden, die Adler usw.) zu eigenen Gruppen [von Lebewesen] zusammenfasste.

Heinz von Foerster, Wahrheit ist die Erfindung eines Lügners, Page 22/23

[Mervis Rosch 1981]
Terminology

ISO 704: Terminology Work: Principles and methods
DIN 2330: Begriffe und Ihre Benennungen

Representation level

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
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</tbody>
</table>

Concept level

Apple:
- Taste
- Color
- Shape

Concept
- attribute a
- attribute b
- attribute c

Object level

Object 1
- property A
- property B
- property C

Object 2
- property A
- property B
- property C

Object 3
- property A
- property B
- property C
Formal Concept Analysis
[Wille 2005]

Models *concepts* as *units of thought*. A concept is constituted by its:

- Extension: consists of all objects belonging to a concept
- Intension: consists of all attributes common to all those objects

Concepts „live“ in relationships with many other concepts where the sub-concept-superconcept-relation plays a prominent role.
Formal Concept Analysis
[Wille 2005]

Formal context:

A Formal Context is a triple $(G, M, I)$ for which $G$ and $M$ are sets while $I$ is a binary relation between $G$ and $M$.

\[ I \subseteq G \times M \]

Formal Concept:

A **formal concept** of a formal context $K := (G,M, I)$ is defined as a pair $(A,B)$ with

\[ A \subseteq G,\quad B \subseteq M \]

and $A = B^\prime$, and $B = A^\prime$; $A$ and $B$ are called the *extent* and the *intent* of the formal concept $(A,B)$, respectively.
Def.: A **formal context** is a triple $(G, M, I)$, where

- $G$ is a set of objects,
- $M$ is a set of attributes
- and $I$ is a relation between $G$ and $M$.

$(g, m) \in I$ is read as „object $g$ has attribute $m$“.
Formal Concept Analysis

Derivation Operators

• $A \subseteq G$, $B \subseteq M$ ($A$...Extent, $B$...Intent)
• $A' := \{m \in M \mid \forall g \in A: (g, m) \in I\}$ all attributes shared by all objects of $A$
• $B' := \{g \in G \mid \forall m \in B: (g, m) \in I\}$ all objects having all attributes of $B$

A formal concept is defined as a pair $(A, B)$

$A = B'$, and $B = A'$

<table>
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<th>Round</th>
<th>Long</th>
<th>Sweet</th>
<th>Sour</th>
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Def.: A formal concept is a pair \((A, B)\), with

- \(A \subseteq G, B \subseteq M\)
  - \(A' := \{m \in M \mid \forall g \in A: (g, m) \in I\}\) all attributes shared by all objects of \(A\)
  - \(B' := \{g \in G \mid \forall m \in B: (g, m) \in I\}\) all objects having all attributes of \(B\)

- \(A' = B\) and \(B' = A\)

Set \(A\) is called the extent (a set of objects)
Set \(B\) is called the intent (a set of attributes)

Of the formal concept \((A, B)\)
Formal Concept Analysis

Sub/Superconcept Relation

- $A \sqsubseteq G, B \sqsubseteq M$
- $A' := \{ m \in M \mid \forall g \in A: (g, m) \in I \}$  all attributes shared by all objects of $A$
- $B' := \{ g \in G \mid \forall m \in B: (g, m) \in I \}$  all objects having all attributes of $B$

The orange concept is a subconcept of the blue concept, since its extent is contained in the blue one. (equivalent to the blue intent is contained in the orange one)
Galois Lattices / Galois Connections
Formal Concept Analysis
Concept Lattices

Where is C1 & C2 located?

Concept Lattices (cf. Galois Lattices)

Formal Concept C1 (A1, A1')

The set of objects that are „yellow“, „sweet“ and „smooth“

Extant

Intent
Concept Lattices (cf. Galois Lattices)

Taste: Sweet/Sour, Shape: Round/Long/, Color: Red/Yellow/..., Texture: Smooth/Bumpy,

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<th>Sour</th>
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Formal Concept Analysis
Concept Lattices
Formal Concept Analysis
Concept Lattices

Def.: The concept lattice of a formal context \((G,M,I)\) is the set of all formal concepts of \((G,M,I)\), together with the partial order

\[(A_1, B_1) \leq (A_2, B_2) :\iff A_1 \subseteq A_2 \quad (\iff B_1 \supseteq B_2).\]

The concept lattice is denoted by \(B(G,M,I)\).

Theorem: The concept lattice is a lattice, i.e. for two concepts \((A_1,B_1)\) and \((A_2,B_2)\), there is always

- a greatest common subconcept \((A_1, B_1) \wedge (A_2, B_2) := (A_1 \cap A_2, (B_1 \cup B_2)''),\)
- and a least common superconcept \((A_1, B_1) \vee (A_2, B_2) := ((A_1 \cup A_2)'', B_1 \cap B_2).\)
Formal Concept Analysis
Greatest Common Subconcept

Which objects share the attributes „smooth“ and „red“ and „sour“?

A: Grapes, Apples

greatest common subconcept (infimum)

- a greatest common subconcept \((A_1, B_1) \land (A_2, B_2) := (A_1 \cap A_2, (B_1 \cup B_2)^\lor)\),
- and a least common superconcept \((A_1, B_1) \lor (A_2, B_2) := ((A_1 \cup A_2)^\land, B_1 \cap B_2)\).
Which attributes share the objects "strawberries" and "lemon"?

A: Bumpy, round

least common superconcept (supremum)

- a greatest common subconcept \((A_1, B_1) \land (A_2, B_2) := (A_1 \cap A_2, (B_1 \cup B_2)^\uparrow)\),
- and a least common superconcept \((A_1, B_1) \lor (A_2, B_2) := ((A_1 \cup A_2)^\downarrow, B_1 \cap B_2)\).
**Def.: An implication**

B1 -> B2 **holds in a context** (G, M, I) if every object intent respects B1 -> B2, i.e.

if each object that has all the attributes in B1 also has all the attributes in B2.

We also say B1->B2 is an implication of (G,M,I).

<table>
<thead>
<tr>
<th>B1</th>
<th>B2</th>
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<tbody>
<tr>
<td>1 &lt; 4 &gt; Smooth ==&gt; Sweet;</td>
<td>10 &lt; 2 &gt; Red Round Sweet Smooth ==&gt; Green Smooth;</td>
</tr>
<tr>
<td>2 &lt; 3 &gt; Red ==&gt; Round Sweet;</td>
<td>11 &lt; 2 &gt; Bumpy ==&gt; Round;</td>
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<tr>
<td>3 &lt; 3 &gt; Yellow Sweet ==&gt; Smooth;</td>
<td>12 &lt; 1 &gt; Round Sour Bumpy ==&gt; Yellow;</td>
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<tr>
<td>4 &lt; 3 &gt; Round Sweet ==&gt; Red;</td>
<td>13 &lt; 0 &gt; Red Green Yellow Round Sweet Sour Smooth Bumpy ==&gt; Long;</td>
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<tr>
<td>5 &lt; 3 &gt; Sour ==&gt; Round;</td>
<td>14 &lt; 0 &gt; Red Green Yellow Round Long Sweet Sour Smooth ==&gt; Bumpy;</td>
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<th>Apple</th>
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</table>
Formal Concept Analysis

Implications

Object count f

1 < 4 > Smooth ==> Sweet,
2 < 3 > Red ==> Round Sweet,
3 < 3 > Yellow Sweet ==> Smooth,
4 < 3 > Round Sweet ==> Red,
5 < 3 > Sour ==> Round,
6 < 2 > Green ==> Red Round Sweet Sour Smooth,
7 < 2 > Yellow Round ==> Sour,
8 < 2 > Long ==> Yellow Sweet Smooth,
9 < 2 > Red Round Sweet Sour ==> Green Smooth,
10 < 2 > Red Round Sweet Smooth ==> Green Sour,
11 < 2 > Bumpy ==> Round,
12 < 1 > Round Sour Bumpy ==> Yellow,
13 < 0 > Red Green Yellow Round Sweet Sour Smooth Bumpy ==> Long,
14 < 0 > Red Green Yellow Round Long Sweet Sour Smooth ==> Bumpy.
Implications:

1. $1 < 8 \rightarrow \text{buy (3557)} \implies \text{get (6562)}$
2. $2 < 4 \rightarrow \text{buy (3557)} \land \text{find (8545)} \implies \text{get (8562)} \implies \text{make (8763)}$
3. $3 < 3 \rightarrow \text{sell (1962)} \implies \text{make (8763)} \land \text{buy (8557)} \implies \text{get (6562)}$
4. $4 < 1 \rightarrow \text{listen (2485)} \implies \text{make (8763)} \land \text{buy (8557)} \land \text{find (8545)} \implies \text{get (6562)} \land \text{do (6391)}$
5. $5 < 1 \rightarrow \text{learn (2014)} \implies \text{make (8763)} \land \text{buy (8557)} \land \text{find (8545)} \implies \text{get (6562)} \land \text{do (6391)}$
6. $6 < 1 \rightarrow \text{play (1598)} \implies \text{make (8763)} \land \text{buy (3557)} \land \text{get (6562)} \land \text{do (6391)} \land \text{sell (1962)}$

Table 7: The 10 most frequent verbs and nouns in the result set and corresponding co-occurrences in queries containing goals.

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<thead>
<tr>
<th>home (2512)</th>
<th>card (2188)</th>
<th>name (1844)</th>
<th>screen (1561)</th>
<th>credit (1433)</th>
<th>music (1398)</th>
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<td>101</td>
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<td>57</td>
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Formal Concept Analysis
Applications / Goal Tagging

Implications:

1 < 2 = find public viewing places in Graz ==> watch the Euro 2008 in Graz
2 < 1 = find a cinema timetable ==> watch a movie in Graz
3 < 1 = find a location on the Graz map ==> find a place located in Graz
4 < 1 = find a specific street in Graz ==> find a place located in Graz
5 < 1 = find cinema program for Graz
6 < 1 = find EM2008 results ==> get detailed information about getting a residential registration form
7 < 1 = find location of Royal English Cinema in Graz
8 < 1 = find English movies to watch in Graz
9 < 1 = find English movies to watch in Graz
10 < 1 = find English movies to watch in Graz
11 < 1 = find information about getting a residential registration form
12 < 1 = find timetable for tram line 7
13 < 1 = get first overview of what Graz offers
14 < 1 = get detailed insights into EURO 2008
15 < 1 = find a place located in Graz
Formal Concept Analysis
Applications / Bugs - Bugzilla

Implications:
FCA / Scaling
Transforming many-valued into single valued contexts

Many-valued contexts

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<th>Color</th>
<th>Shape</th>
<th>Taste</th>
<th>Texture</th>
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<td>Sweet/sour</td>
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What hasn’t been mentioned yet

Via Scales

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<td></td>
</tr>
<tr>
<td>Grapes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pear</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Knowledge Acquisition
CIMIANO, HOTHO, & STAAB 2005

Figure 1: Overall Process
Knowledge Acquisition
CIMIANO, HOTHO, & STAAB 2005

Figure 2: The lattice of formal concepts (left) and the corresponding hierarchy of ontological concepts (right) for the tourism example.
Knowledge Acquisition
CIMIANO, HOTHO, & STAAB 2005

CIMIANO, HOTHO, & STAAB

Figure 3: Examples of lattices automatically derived from tourism-related texts without smoothing (left) and with smoothing (right)
Figure 4: Example for an automatically acquired concept hierarchy $O_{auto}$ (left) compared to the reference concept hierarchy $O_{ref}$ (right).

Figure 5: Example for a perfectly learned concept hierarchy $O_{perfect}$ (left) compared to the reference concept hierarchy $O_{ref}$ (right).
How to compute/draw a concept lattice (manually):

- From left to right, consider all intersections of each column extent with every column extent to the left of it. If the resulting extent is not already a column, add it as column at the right end of the context. Repeat this until the last (added) column is reached.
- Add a full column, unless there is already one. (Now each column stands for one concept.)
- Draw a circle for the full column.
- Draw for each column, starting for the ones with a maximal number of crosses, a circle, and link it with a line to the circles where the column comprises the current column.
- Attach every attribute label to the circle of the corresponding column.
- Attach every object label to the circle lying exactly below the circles of the attributes in its intent.

How to check the drawing of a concept lattice:

- Is it really a lattice? (This test is usually skipped.)
- Is every concept with exactly one upper neighbor labeled by at least one attribute?
- Is every concept with exactly one lower neighbor labeled by at least one object?
- Is, for all $g \in G$ and all $m \in M$, the label of object $g$ below the label of attribute $m$ iff $(g,m) \in I$? (This test is covered when you manage to place all objects correctly.)
Instruction: how to determine all formal concepts of a small formal context

1. Initialize a list of concept extents. To begin with, write for each attribute \( m \in M \) the attribute extent \( \{m\}' \) to this list (if not already present).

2. For any two sets in this list, compute their intersection. If the result is a set that is not yet in the list, then extend the list by this set. With the extended list, continue to build all pairwise intersections.

3. If for any two sets of the list their intersection is also in the list, then extend the list by the set \( G \) (provided it is not yet contained in the list). The list then contains all concept extents (and nothing else).

4. For every concept extent \( A \) in the list compute the corresponding intent \( A' \) to obtain a list of all formal concepts \( (A, A') \) of \( (G, M, I) \).

Ganter / Stumme 2003
### Knowledge Management Institute

Markus Strohmaier 2012

#### Formal Concept Analysis

**Lattice Construction Algorithms**

**Example:**

1. Write the attribute extents to a list.

<table>
<thead>
<tr>
<th>No.</th>
<th>extent</th>
<th>found as</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁</td>
<td>{T₄}</td>
<td>{a}'</td>
</tr>
<tr>
<td>e₂</td>
<td>{T₁, T₂, T₄, T₆}</td>
<td>{b}'</td>
</tr>
<tr>
<td>e₃</td>
<td>{T₃, T₄, T₆}</td>
<td>{c}'</td>
</tr>
<tr>
<td>e₄</td>
<td>{T₁, T₅}</td>
<td>{d}'</td>
</tr>
<tr>
<td>e₅</td>
<td>{T₂, T₇}</td>
<td>{e}'</td>
</tr>
</tbody>
</table>

2. Compute all pairwise intersections, and

3. add G

<table>
<thead>
<tr>
<th>No.</th>
<th>extent</th>
<th>found as</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁</td>
<td>{T₄}</td>
<td>{a}'</td>
</tr>
<tr>
<td>e₂</td>
<td>{T₁, T₂, T₄, T₆}</td>
<td>{b}'</td>
</tr>
<tr>
<td>e₃</td>
<td>{T₃, T₄, T₆}</td>
<td>{c}'</td>
</tr>
<tr>
<td>e₄</td>
<td>{T₁, T₅}</td>
<td>{d}'</td>
</tr>
<tr>
<td>e₅</td>
<td>{T₂, T₇}</td>
<td>{e}'</td>
</tr>
<tr>
<td>e₆</td>
<td>{}</td>
<td>e₁ ∩ e₄</td>
</tr>
<tr>
<td>e₇</td>
<td>{T₄, T₆}</td>
<td>e₂ ∩ e₃</td>
</tr>
<tr>
<td>e₈</td>
<td>{T₁}</td>
<td>e₂ ∩ e₄</td>
</tr>
<tr>
<td>e₉</td>
<td>{T₂}</td>
<td>e₂ ∩ e₅</td>
</tr>
<tr>
<td>e₁₀</td>
<td>{T₁, T₂, T₃, T₄, T₅, T₆, T₇}</td>
<td>step 3</td>
</tr>
</tbody>
</table>

4. Compute the intents

(Formal Concepts)

<table>
<thead>
<tr>
<th>Concept No.</th>
<th>(extent, intent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{(T₄), {a, b, c}}</td>
</tr>
<tr>
<td>2</td>
<td>{(T₁, T₂, T₄, T₆), {b}}</td>
</tr>
<tr>
<td>3</td>
<td>{(T₃, T₄, T₆}, {c}}</td>
</tr>
<tr>
<td>4</td>
<td>{(T₇, T₅}, {d}}</td>
</tr>
<tr>
<td>5</td>
<td>{(T₂, T₇}, {e}}</td>
</tr>
<tr>
<td>6</td>
<td>{}, {a, b, c, d, e}}</td>
</tr>
<tr>
<td>7</td>
<td>{(T₄, T₆}, {b, c}}</td>
</tr>
<tr>
<td>8</td>
<td>{(T₁}, {b, d}}</td>
</tr>
<tr>
<td>9</td>
<td>{(T₂}, {b, e}}</td>
</tr>
<tr>
<td>10</td>
<td>{(T₁, T₂, T₃, T₄, T₅, T₆, T₇}, {0}}</td>
</tr>
</tbody>
</table>

Ganter / Stumme 2003
Formal Concept Analysis
Lattice Construction Algorithms

A Naive Approach:

Instruction how to draw a line diagram of a small concept lattice

5. Take a sheet of paper and draw a small circle for every formal concept, in the following manner: a circle for a concept is always positioned higher than the all circles for its proper subconcepts.

6. Connect each circle with the circles of its lower neighbors.

7. Label with attribute names: attach the attribute $m$ to the circle representing the concept ($\{m\}', \{m\}'$).

8. Label with object names: attach each object $g$ to the circle representing the concept ($\{g\}'$, $\{g\}'$).

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5. Draw a circle for each of the formal concepts:

6. Connect circles with their lower neighbours.

Why is there no line between FC4 & 7?

A circle for a concept is always positioned higher than all circles for its proper subconcepts.

Definition 14 Let \( (A_1, B_1) \) and \( (A_2, B_2) \) be formal concepts of \( (G, M, I) \).

We say that \( (A_1, B_1) \) is a proper subconcept of \( (A_2, B_2) \) if \( (A_1, B_1) < (A_2, B_2) \) and, in addition, \( (A_1, B_1) \neq (A_2, B_2) \) holds. As an abbreviation, we write \( (A_1, B_1) < (A_2, B_2) \).

We say that \( (A_1, B_1) \) is a lower neighbour of \( (A_2, B_2) \), if \( (A_1, B_1) < (A_2, B_2) \), but no formal concept \( (A, B) \) of \( (G, M, I) \) exists with \( (A_1, B_1) < (A, B) < (A_2, B_2) \).

The abbreviation for this is \( (A_1, B_1) \prec (A_2, B_2) \).
Formal Concept Analysis
Lattice Construction Algorithms

Example ctd`:

7. Add the attribute names:

8. Determine the object concepts

<table>
<thead>
<tr>
<th>Object</th>
<th>object intent ([g]')</th>
<th>No. of concept</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_1)</td>
<td>{b, d}</td>
<td>8</td>
</tr>
<tr>
<td>(T_2)</td>
<td>{b, e}</td>
<td>9</td>
</tr>
<tr>
<td>(T_3)</td>
<td>{c}</td>
<td>3</td>
</tr>
<tr>
<td>(T_4)</td>
<td>{a, b, c}</td>
<td>1</td>
</tr>
<tr>
<td>(T_5)</td>
<td>{d}</td>
<td>4</td>
</tr>
<tr>
<td>(T_6)</td>
<td>{b, c}</td>
<td>7</td>
</tr>
<tr>
<td>(T_7)</td>
<td>{c}</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_1)</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T_2)</td>
<td>×</td>
<td></td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>(T_3)</td>
<td>×</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>(T_4)</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T_5)</td>
<td></td>
<td>×</td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>(T_6)</td>
<td>×</td>
<td>×</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(T_7)</td>
<td></td>
<td></td>
<td>×</td>
<td>×</td>
</tr>
</tbody>
</table>

Label with attribute names: attach the attribute \(m\) to the circle representing the concept \((\{m\}' , \{m\}''\).
Formal Concept Analysis

Tools:
E.g.
- JaLaBa [http://maarten.janssenweb.net/jalaba/JaLaBA.pl](http://maarten.janssenweb.net/jalaba/JaLaBA.pl)

Further Information
- FCA Homepage [http://www.upriss.org.uk/fca/fca.html](http://www.upriss.org.uk/fca/fca.html)
Formal Concept Analysis

• ConExp http://conexp.sourceforge.net/index.html
Bonus Task

• Sketch a categorization example
• Define a Formal Context, for which $|G| \geq 10$, $|M| \geq 10$ and $|I| \sim |G| + |M|
• Use Conexp to
  – Name all elements of G and M (choose plausible G and M)
  – Represent your formal context (choose plausible I)
  – Draw the Concept Lattice
  – Calculate Implications (there should be at least one implication with $f > 3$)
• Submit
  – A one-page .pdf file that contains your 1) context, 2) the laid out(!) lattice 3) the top 10 implications and 4) a brief interpretation of your example
    • Name the pdf File using the following Syntax: „GWM12-BT1-YOURMATR-YOURLASTNAME.pdf“
    – To me via e-mail using subject „[GWM12-BT1-YOURMATR]“
    – before the beginning of next week ‘s class
Any questions?

See you next week!